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DYNAMIC PROGRAMMING AND
MEAN SQUARE DEVIATION

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SUMMARY

In this paper we wish to discuss some applications of the functional equation technique of dynamic programming to the treatment of some quadratic variational problems and the linear equations arising therefrom.

The first problem we shall consider is that of determining the minimum value of the quadratic deviation

$$\int_0^T (f - \sum_{k=1}^N x_k\phi_k)^2 dx,$$

where $f(x)$ is a given function of $\{\phi_k(x)\}$, a given sequence of real functions.

Next we consider the problem of minimizing the quadratic form

$$Q_{N,M}(x) = \sum_{k=0}^N (b_k - \sum_{l=0}^M x_l a_{k-l})^2$$

over all real x_i, where $\{a_k\}$ and $\{b_k\}$ are given real sequences. This problem arises in the Kolmogoroff-Wiener theory of linear predictors, and the limiting verrion of the problem, $N = \infty$, is discussed by Levinson in the appendix to Wiener's book.

Finally we discuss the problem of solving the linear system

$$Ax = b,$$

under the assumption that A is positive definite.

Although it would seem that in all three cases the methods we present have some computational utility, we shall not discuss these matters here, since at the moment we are interested only in the purely analytic aspects of these questions.

DYNAMIC PROGRAMMING AND MEAN SQUARE DEVIATION

Richard Bellman

§1. INTRODUCTION

In this paper we wish to discuss some applications of the functional equation technique of dynamic programming [1] to the treatment of some quadratic variational problems and the linear equations arising therefrom.

The first problem we shall consider is that of determining the minimum value of the quadratic deviation

$$(1) \qquad \int_0^T \left(f - \sum_{k=1}^{N} x_k \phi_k\right)^2 dx,$$

where $f(x)$ is a given function of $\{\phi_k(x)\}$, a given sequence of real functions.

Next we consider the problem of minimizing the quadratic form

$$(2) \qquad Q_{N,M}(x) = \sum_{k=0}^{N} \left(b_k - \sum_{\ell=0}^{M} x_\ell a_{k-\ell}\right)^2$$

over all real x_1, where $\{a_k\}$ and $\{b_k\}$ are given real sequences. This problem arises in the Kolmogoroff-Wiener theory of linear predictors, and the limiting version of the problem, $N = \infty$, is discussed by Levinson in the appendix to Wiener's book, [2].

Finally we discuss the problem of solving the linear system

(3) $\quad Ax = b,$

under the assumption that A is positive definite.

Although it would seem that in all three cases the methods we present have some computational utility, we shall not discuss these matters here, since at the moment we are interested only in the purely analytic aspects of these questions.

§2. QUADRATIC DEVIATION

The problem of minimizing the quadratic form

$$Q_N(u) = \int_0^T \left(f - \sum_{k=1}^N u_k \phi_k\right)^2 dx \tag{1}$$

is, as we know, quite easily resoluble. Let $\{\psi_k\}$ denote the orthonormal sequence formed from $\{\phi_k\}$ by means of the Gram-Schmidt orthogonalization procedure. The functions $\phi_k(x)$ are assumed to be linearly independent. Then

$$\begin{aligned} \min_u Q_N(u) &= \min_y \int_0^T \left(f - \sum_{k=1}^N y_k \psi_k\right)^2 dx \\ &= \int_0^T f^2 dx - \sum_{k=1}^N \left(\int_0^T f\psi_k dx\right)^2 \qquad (2) \\ &= \int_0^T f^2 dx - \int_0^T \int_0^T f(x)f(y)K_N(x,y)dxdy, \end{aligned}$$

where

$$K_N(x,y) = \sum_{k=1}^N \psi_k(x)\psi_k(y). \tag{3}$$

We wish to obtain a recurrence relation for $K_N(x,y)$ without going through the orthogonalization procedure. To do this, we introduce the quadratic functional

$$(4) \qquad F_N(f) = \min_u \int_0^T \left(f - \sum_{k=1}^N u_k \phi_k\right)^2 dx, \quad N = 1, 2, \dots .$$

Then, using the principle of optimality,

$$(5) \qquad F_N(f) = \min_{u_N} F_{N-1}(f - u_N \phi_N), \quad N = 2, 3, \dots .$$

To utilize (5), we use the known form of $F_N(f)$ obtained in (2) above. We have

$$F_N(f) = \min_{u_N} \left[\int_0^T (f - u_N\phi_N)^2 dx - \int_0^T \int_0^T (f(x) - u_N\phi_N(x))(f(y) - u_N\phi_N(y)) K_{N-1}(x,y)\,dx\,dy \right]$$

$$(6) \qquad = \min_{u_N} \left[\int_0^T f^2 dx - \int_0^T \int_0^T K_{N-1}(x,y) f(x) f(y)\,dx\,dy - 2u_N \left\{ \int_0^T f\phi_N dx - \int_0^T \int_0^T \phi_N(x) f(y) K_{N-1}(x,y)\,dx\,dy \right\} + u_N^2 \left\{ \int_0^T \phi_N^2 dx - \int_0^T \int_0^T K_{N-1}(x,y)\phi_N(x)\phi_N(y)\,dx\,dy \right\} \right].$$

It is easily seen that the coefficient of u_N^2 is positive. The minimizing value is given by

$$(7) \qquad u_N = \frac{\int_0^T f\phi_N dx - \int_0^T \int_0^T \phi_N(x) f(y) K_{N-1}(x,y)\,dx\,dy}{D_N},$$

where, to simplify the notation, we have set

(8) $$D_N = \int_0^T \phi_N^2 dx - \int_0^T \int_0^T K_{N-1}(x,y)\phi_N(x)\phi_N(y)dxdy.$$

The minimum value is given by

(9) $$F_N(f) = \left[\int_0^T f^2 dx - \int_0^T \int_0^T K_{N-1}(x,y)f(x)f(y)dxdy\right] - \left[\int_0^T f\phi_N dx - \int_0^T \int_0^T \phi_N(x)f(y)K_{N-1}(x,y)dxdy\right]^2 \Big/ D_N.$$

To obtain the desired relation between K_N and K_{N-1}, we compare (9) and (2), and equate coefficients of $f(x)f(y)$. Cancelling the term $\int_0^T f^2 dx$, the result is

$$K_N(x,y) = K_{N-1}(x,y) + \frac{\phi_N(x)\phi_N(y)}{D_N} - \frac{2\int_0^T \phi_N(z)K_{N-1}(z,y)dzdy}{D_N} + \frac{1}{D_N}\int_0^T \int_0^T \phi_N(z)\phi_N(w)K_{N-1}(z,x)K_{N-1}(w,y)dzdw,$$

a nonlinear integral equation.

§3. LINEAR PREDICTORS

Consider the quadratic form

(1) $$Q_{N,M}(x) = \sum_{k=0}^{N} (b_k - \sum_{\ell=0}^{M} x_\ell a_{k-\ell})^2, \quad N > M \geq 1,$$

and the sequence defined by

(2) $$f_{N,M} = \operatorname*{Min}_x Q_{N,M}(x).$$

We wish to determine a relation between $f_{N,M}$ and $f_{N-1,M-1}$ which will enable us to compute $f_{N,M}$ starting with $f_{N-M+1,1}$.

Consider the function of $N + 1$ real variables $y_0, y_1, \ldots, y_N$, and the integers N and M defined by

$$(3)\qquad f_{N,M}(y_0,y_1,\ldots,y_N) = \underset{x}{\mathrm{Min}}\,[(y_0-x_0a_0)^2 + (y_1-x_0a_1-x_1a_0)^2 + \cdots + (y_N-x_0a_N-x_1a_{N-1}-\cdots-x_Ma_{N-M})^2],$$

where the sequences $\{y_i\}$ and $\{a_i\}$ are given and the minimization is over the x_i. In order for the problem to be non-trivial, we assume that $N > M \geq 1$.

Let us now obtain a recurrence relation. Assume that x_0 has been chosen. Then the quantities $x_1, x_2, \ldots, x_M$ are to be chosen to minimize the remaining sum

$$(4)\qquad (y_1-x_0a_1-x_1a_0)^2 + (y_2-x_0a_2-x_1a_1-x_2a_0)^2 + \cdots + (y_N-x_0a_N-x_1a_{N-1}-\cdots-x_Ma_{N-M})^2.$$

This minimum is precisely, in accordance with the notation introduced above,

$$(5)\qquad f_{N-1,M-1}(y_1-x_0a_1,\ y_2-x_0a_2,\ \ldots,\ y_N-x_0a_N).$$

Hence we obtain the recurrence relation

$$f_{N,M}(y_0,y_1,\ldots,y_N) = \operatorname*{Min}_{x_0} [(y_0-x_0a_0)^2 + \tag{6}$$
$$f_{N-1,M-1}(y_1-x_0a_1,y_2-x_0a_2,\ldots,y_N-x_0a_N)].$$

For $M = 1$, we have

$$f_{N,1}(y_0,y_1,\ldots,y_N) = \operatorname*{Min}_{x_0} [\sum_{k=0}^{N} (y_k-x_0a_k)^2] \tag{7}$$
$$= (\sum_{k=0}^{N} y_k^2)(\sum_{k=0}^{N} a_k^2) - (\sum_{k=0}^{N} a_ky_k)^2.$$

§4. RECURRENCE RELATIONS

In order to use (3.5) to determine the functions $f_{N,M}(y)$ in a more constructive fashion, let us observe that these functions are quadratic forms in the y_i,

$$f_{N,M}(y_0,y_1,\ldots,y_N) = \sum_{i,j=0}^{N} c_{ij}(N,M)y_iy_j. \tag{1}$$

Substituting in (3.5), we obtain the result

$$\sum_{i,j=0}^{N} c_{ij}(N,M)y_iy_j = \operatorname*{Min}_{x_0} [(y_0-x_0a_0)^2 + \tag{2}$$
$$\sum_{i,j=0}^{N-1} c_{ij}(N-1,M-1)(y_{i+1}-x_0a_i)(y_{j+1}-x_0a_j)].$$

Hence x_0 is determined by the relation

$$x_0 = \frac{a_0y_0 + \sum_{i,j=0}^{N-1} c_{ij}(N-1,M-1)(y_{i+1}a_j + y_{j+1}a_i)/2}{\sum_{i,j=0}^{N-1} c_{ij}(N-1,M-1)a_ia_j}. \tag{3}$$

and we have the relation

$$\sum_{i,j=0}^{N} c_{ij}(N,M)y_i y_j = \left\{y_0^2 + \sum_{i,j=0}^{N-1} c_{ij}(N-1,M-1)y_{i+1}y_{j+1}\right\}$$

(4)
$$\left\{\sum_{i,j=0}^{N-1} c_{ij}(N-1,M-1)a_i a_j\right\}$$

$$-\left\{a_0 y_0 + \sum_{i,j=0}^{N-1} c_{ij}(N-1,M-1)\right.$$

$$\left.(y_{i+1}a_j + y_{j+1}a_i)/2\right\}^2.$$

Equating coefficients of $y_i y_j$ on both sides of this equation, we will obtain a relation for $c_{ij}(N,M)$ in terms of the set $\{c_{ij}(N-1,M-1)\}$. Once we have obtained the $c_{ij}(N,M)$, we can then calculate the elements of the minimizing sequence $\{x_i\}$, using (3) repeatedly.

§5. THE EQUATION $Ax = b$

Let us now sketch the application of the same techniques to the problem of solving the system of equations

(1) $$\sum_{j=1}^{N} a_{ij}x_j = y_i, \quad i = 1, 2, \ldots, N,$$

where $A = (a_{ij})$ is a positive definite matrix.

Introduce the function of N variables, $f_N(y_1,y_2,\ldots,y_N)$, for $N = 1, 2, \ldots, -\infty < y_i < \infty$, by means of the relation

(2) $$f_N(y_1,y_2,\ldots,y_N) = \operatorname*{Min}_{x_i} \left[\sum_{i,j=1}^{N} a_{ij}x_i x_j - 2\sum_{i=1}^{N} x_i y_i\right].$$

To obtain an equation for f, we write

(3)
$$\sum_{i,j=1}^{N} a_{ij}x_ix_j - 2\sum_{i=1}^{N} x_iy_i = a_{NN}x_N^2 + \sum_{i,j=1}^{N-1} a_{ij}x_ix_j - 2\sum_{i=1}^{N-1} x_i(y_i - a_{iN}x_N) - 2x_Ny_N.$$

Once x_N has been chosen, it is clear that the remaining x_i, $i = 1, 2, \ldots, N-1$, will be chosen to minimize the expression

(4)
$$\sum_{i,j=1}^{N-1} a_{ij}x_ix_j - 2\sum_{i=1}^{N-1} x_i(y_i - a_{iN}x_N).$$

In accordance with the above notation, this minimum is precisely

(5)
$$f_{N-1}(y_1 - a_{1N}x_N,\ y_2 - a_{2N}x_N,\ \ldots,\ y_{N-1} - a_{N-1,N}x_N).$$

Hence, we obtain the recurrence relation

(6)
$$f_N(y_1, y_2, \ldots, y_N) = \min_{x_N} [a_{NN}x_N^2 + f_{N-1}(y_1 - a_{1N}x_N,\ y_2 - a_{2N}x_N,\ \ldots,\ y_{N-1} - a_{N-1,N}x_N) - 2\,x_Ny_N].$$

In order to use this relation constructively, we use the fact that f_N is a quadratic form in the y_i,

(7)
$$f_N(y_1, y_2, \ldots, y_N) = \sum_{i,j=1}^{N} c_{ij}(N)y_iy_j,$$

for $N = 1, 2, \ldots$.

Returning to equation (6), we have the relation

$$\sum_{i,j=1}^{N} c_{ij}(N)y_i y_j = \underset{x_N}{\mathrm{Min}} \Big[a_{NN}x_N^2 - 2x_N y_N \tag{8}$$
$$+ \sum_{i,j=1}^{N-1} c_{ij}(N-1)(y_i + a_{iN}x_N)(y_j + a_{jN}x_N)\Big].$$

Collecting terms on the right, we have

$$\sum_{i,j=1}^{N} c_{ij}(N)y_i y_j = \underset{x_N}{\mathrm{Min}} \Big[x_N^2 \Big\{a_{NN} + \sum_{i,j=1}^{N-1} a_{iN}a_{jN}c_{ij}(N-1)\Big\} +$$
$$x_N \Big\{-2y_N + \sum_{i,j=1}^{N-1} [y_i a_{jN} + y_j a_{iN}] c_{ij}(N-1)\Big\} \tag{9}$$
$$+ \sum_{i,j=1}^{N-1} c_{ij}(N-1)y_i y_j \Big].$$

The minimization can now be performed readily and the recurrence relations connecting $\{c_{ij}(N)\}$ with $\{c_{ij}(N-1)\}$ read off.

§6. DISCUSSION

The method discussed above might be particularly useful in connecting with the problem of solving an infinite system of equations of the form

$$\sum_{j=1}^{\infty} a_{ij}x_j = b_i, \quad i = 1, 2, \ldots, \tag{1}$$

where we solve successively the finite systems

$$\sum_{j=1}^{N} a_{ij}x_j = b_i, \quad i = 1, 2, \ldots, N. \tag{2}$$

to obtain approximations to the solutions of (1).

REFERENCES

1. R. Bellman, _Dynamic Programming_, Princeton University Press, 1957.

2. N. Wiener, _Extrapolation, Interpolation and Smoothing of Stationary Time Series_, Technology Press and J. Wiley, 1949.